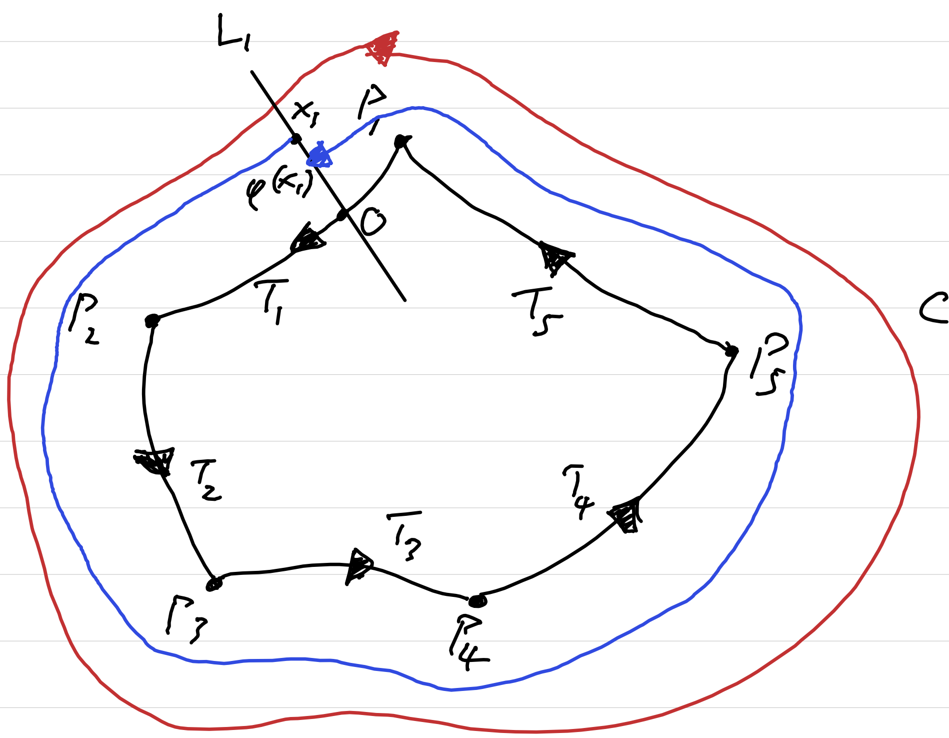
$\xi : \mathbb{R}^2 \to \mathbb{R}^2$ real analytic vector field

Dulac's problem: If ξ has a r.a. continuation on S^2, then ξ has finitely many limit cycles.

$\rightarrow$ open (again...)

The main situation to be addressed:



P_i : elementary singularity of ξ

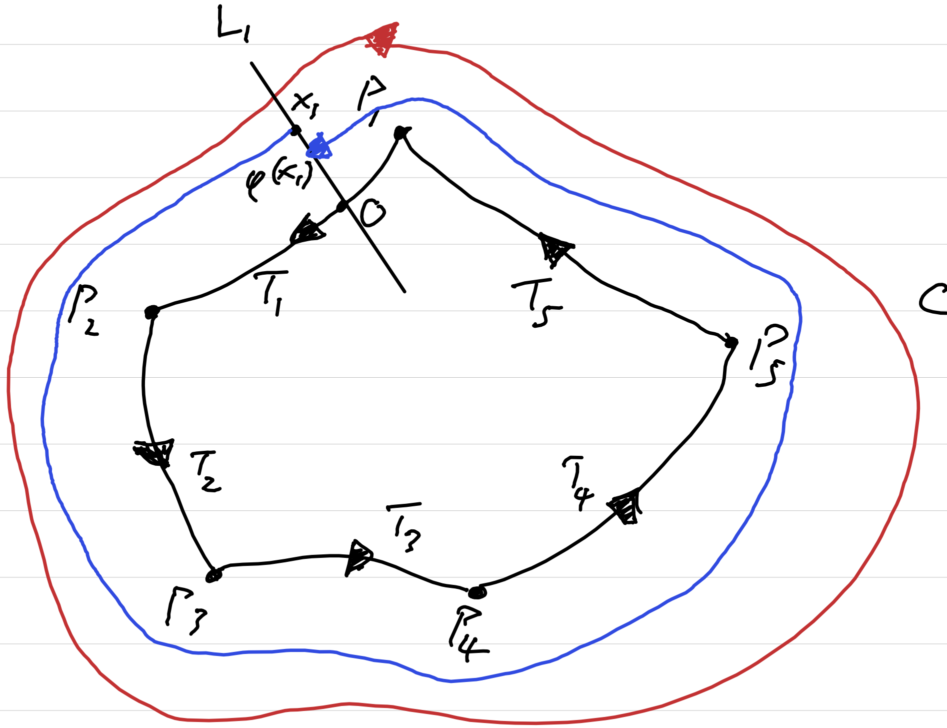
T_i : trajectory connecting P_i to P_{i+1} (convention: $P_6 = P_1$)

polycycle Γ : simple closed curve formed by the above

C : a nearby cycle of ξ

L_1 : segment transverse to ξ with coordinate x_1 (equal to 0 at $L_1 \cap T_1$)

φ : Poincaré first-return map that maps $x_1 \in L$ to the first intersection $\varphi(x_1) \in L$ of the trajectory through x_1 with L.



<u>Then:</u> $x_1 \in L$ lies on a limit cycle iff x_1 is an isolated fixed point of φ.

One needs to show that φ does not have isolated fixed points piling up towards $x_1 = 0$.

<u>Goal:</u> show that φ belongs to a Hardy field.

Expl.: If Γ is a cycle, then by an old theorem of Poincaré's the map φ is real analytic at 0
$\Longrightarrow$ φ has finitely many isolated fixed points near $x_1 = 0$, or $\varphi \equiv id$.

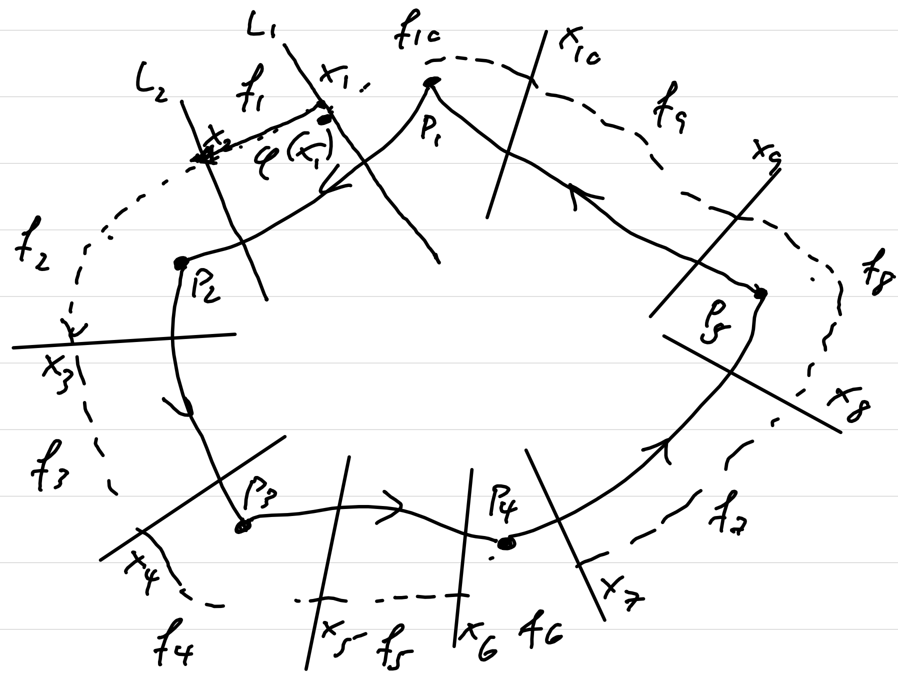
In the example, φ belongs to the Hardy field L of germs at 0^+ of all sums of convergent real Laurent series.

We know that L is a Hardy field because the Taylor expansion map $T : L \to \mathbb{R}((x))$ is injective

$\longrightarrow$ (L, T) is a quasianalytic field

Back to the polycycle Γ: in general φ is not real analytic at 0.

To study φ, we decompose it into transition maps:



$$\varphi(x_1) = f_{10} \circ \cdots \circ f_1(x_1)$$

where

$$f_i \text{ is } \begin{cases} \bullet \text{ real analytic if } i \text{ is odd} \\ \bullet \text{ not real analytic if } i \text{ is even.} \end{cases}$$

<u>New Goal:</u> Show there is a Hardy field $\mathcal{K}_{trans}$ that contains all transition maps near el. sing. of planar r.a. vector fields, and that is closed under composition.

<u>Remark:</u> If we can find a Hardy field $\mathcal{K}$ containing all such transition maps such that the expansion $\mathbb{R}_{trans}$ of the real field by $\mathcal{K}$ is o-minimal, then take $\mathcal{K}_{trans}$ = Hardy field of all defbl germs of $\mathbb{R}_{trans}$.

There are two types of transition maps $f: (0,\varepsilon) \to (0,\varepsilon)$:

① if f is near a hyperbolic sing. of ξ, then f is "almost regular"

② if f is near an elem. non-hyp. sing. of ξ, then f is "almost

Definition ① (Ilyashenko)

$f : (0,\varepsilon) \to (0,\varepsilon)$ is almost regular if there exist

(i) $0 < \alpha_0 < \alpha_1 < \dots \longrightarrow \infty$

(ii) $p_0 \in \mathbb{R}^\times$ and $p_i \in \mathbb{R}[x]$ for $i > 0$

(iii) $C > 0$ and a holomorphic $\bar{f} : \Omega_C \to \mathbb{C}$, where

$$\Omega_C := \{ z + C\sqrt{1+z} : \mathrm{Re}(z) > 0 \}$$

is a standard quadratic domain

such that $\bar{f}\restriction_{\mathbb{R}} = f \circ e^{-x}$ and for all $n \in \mathbb{N}$,

$$\bar{f}(z) - \sum_{i=0}^{n} p_i(z) e^{-\alpha_i z} = o\left(e^{-\alpha_n \mathrm{Re}(z)}\right)$$

as $\mathrm{Re}(z) \to \infty$ in Ω_C.

The series $\hat{f}(x) = \sum_{i=0}^{\infty} p_i(x) e^{-\alpha_i x}$ is called a Dulac series.

Expl: if $f \in L$, then f is almost regular with $\hat{f} = (T_0 f) \circ e^{-x}$, which converges on some right half-plane. In particular, all $\rho_i \in \mathbb{R}$ in this case.

In general, ρ_i is not constant and the series $\hat{f}$ diverges.

Peculiarities: the composition of two a.r. maps is a.r.

However, sums of a.r. maps need not be a.r.!

Uniqueness Principle (Ilyashenko)
If $g : \Omega_C \to \mathbb{C}$ is bounded and holom and if $g(z) = o(e^{-n \mathrm{Re}(z)})$ as $\mathrm{Re}(z) \to \infty$ in Ω_C, for all $n \in \mathbb{N}$, then $g \equiv 0$.

$\Longrightarrow$ the set $\mathcal{A}$ of all a.r. germs is quasianalytic.

If we drop the requirement that $p_0 \in \mathbb{R}^\times$, we would get a q.a. algebra instead, but it would not be closed under composition.

Instead, take $\mathcal{A}_0$ to be all $f : (a, \infty) \to \mathbb{R}$ for which there exist

- $0 < \alpha_0 < \alpha_1 < \cdots \nearrow \infty$
- $c_i \in \mathbb{R}$
- a holom. cont. $\tilde{f} : \Omega_c \to \mathbb{C}$ of f

such that $\tilde{f}(z) \sim \sum c_i e^{-\alpha_i z}$ as $\mathrm{Re}(z) \to \infty$ in Ω_c.

UP $\Rightarrow$ $\mathcal{A}_0$ is a q.a. algebra.

$\Rightarrow$ field of fractions $\mathcal{K}_0$ is q.a. as well.

Note: $\mathcal{K}_0 \circ \log$ is a q.a. field too, and each $f \in \mathcal{K}_0$ is log-bounded

So take $\mathcal{A}_1$ to be all $f:(a,\infty) \to \mathbb{R}$ for which there exist
- $0 < \alpha_0 < \alpha_1 < \dots \to \infty$
- $c_i \in \mathcal{K}_0 \circ \log$
- a holom. cont. $\tilde{f} : \Omega_c \to \mathbb{C}$ of f

such that $\tilde{f}(z) \sim \sum c_i(z) e^{-\alpha_i z}$ as $\mathrm{Re}(z) \to \infty$ in Ω_c.

UP $\Rightarrow$ $\mathcal{A}_1$ is a q.a. algebra.

$\Rightarrow$ field of fractions $\mathcal{K}_1$ is q.a. as well.

Note: $\mathcal{K}_0 \subseteq \mathcal{K}_1$ and $\mathcal{K}_0 \circ \log \subseteq \mathcal{K}_1$.

$\Rightarrow$ Iterate by induction on $k \in \mathbb{N}$, then set

$$\mathcal{K}_{ar} := \bigcup_{k \in \mathbb{N}} \mathcal{K}_k .$$

Theorem (2018): $\mathcal{K}_{ar}$ is closed under differentiation and log-composition, i.e., if $f, g \in \mathcal{K}_{ar}$ and $g(x) \to \infty$ as $x \to \infty$, then $f \circ \log \circ g \in \mathcal{K}_{ar}$.

$\Rightarrow$ $\mathcal{K}_{ar} \circ (-\log)$ closed under composition

Not-quite-

<u>Definition</u> ② (Ramis, Sibuya, Tougeron)

$f : (\mathbb{C}, \mathcal{E}) \to (\mathbb{C}, \mathcal{E})$ is <u>k-summable</u> (in the pos. real direction) if $k > 0$, there exist a strip

$$S = S(R, \alpha) = \{z \in \mathbb{C} : \mathrm{Re}(z) > R, |\mathrm{Im}(z)| < \alpha\}$$

with $\alpha > \frac{\pi}{2k}$

and, for each $p \in \mathbb{N}$, a half-plane

$$H_p = \{z \in \mathbb{C} : \mathrm{Re}(z) > R_p\}$$

with $R = R_0 < R_1 < \cdots \to \infty$ depending on k

and a holom. function $f : S_p \to \mathbb{C}$, where

$$S_p := S \cup H_p,$$

such that each f_p is given by a convergent power series in e^{-z}, $\sum_p f_p$ converges unif. on compact subsets of S to a function $\bar{f} : S \to \mathbb{C}$ that extends f, and

$$\|\bar{f}\| := \sum_p r^p \|f_p\|_{S_p} < \infty$$

for some $r > 0$.

f is <u>multisummable</u> if it is a finite sum of k-summable functions, for various k.

<u>Fact (Ramis–Sibuya)</u>: The set of multisummable germs at $+\infty$ is a q.a. algebra.

$\Rightarrow$ it's fraction field $\mathcal{G}$ is a Hardy field that embeds in $\mathbb{R}((x))$.

<u>Example</u>: Let
$$\varphi(z) := \log\Gamma(z) - \left(z - \tfrac{1}{2}\right)\log z + z - \frac{\log 2\pi}{2}.$$
Then $\varphi \circ e^{-z}$ is 1-summable, and its Taylor series at ∞ is divergent.

How can we combine $\mathcal{K}_{an}$ and $\mathcal{G}$?

Problems: not the same type of series...

Theorem (Rolin, Servi, S)

If, in the definition of multisummable we replace the f_p by functions defined by convergent gen. power series in e^{-x}, we obtain the new q.a. field of multisummable generalized germs, $\mathcal{G}^*$.

$\mathcal{G}^*$ contains both multisummable germs and germs defined by convergent generalized power series.

Note: $\mathcal{A}_0$ contains all germs def. by conv. gen. power series.

Idea (Sec-5) Replace H_p by standard quadratic domain
$$W_p = \{z + C_p\sqrt{1+z} : \mathrm{Re}(z) > 0\}$$
with $C_p \nearrow \infty$, and replace f_r by a.r. function on
$$S_p := S \cup W_p.$$

Theorem (Sec): Under the right regularity conditions on $\{f_r\}$, we obtain a notion of multisummable germs over $\mathcal{A}_c$. The set of all such germs is a q.a. field $\hat{\mathcal{F}}_0$ that extends both $\mathcal{A}_c$ and $\mathcal{G}$, (as well as $\mathcal{G}^*$).

Note: $\mathcal{F}_0 \circ \log$ is a polynomially bounded q.a. field.

Repeating the construction of $\mathcal{A}_1$, but with $\mathcal{F}_0$ in place of $\mathcal{K}_0$, we obtain a q.a. field $\tilde{\mathcal{K}}_1$.

<u>Quiz question:</u> $\mathcal{E}$ is $\mathcal{G} \subseteq \tilde{\mathcal{K}}_1$?

No: only $\mathcal{G} \circ \log \subseteq \tilde{\mathcal{K}}_1$.

<u>Next step:</u> define multisum. over $\tilde{\mathcal{K}}_1$.