Representation Type, Decidability and Pseudofinite-dimensional Modules over Finite-dimensional Algebras

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Prest's Conjecture (Mid 80s)

Let $\mathcal{A}$ be a finite-dimensional algebra over a recursive field. The theory of $\mathcal{A}$-modules is undecidable if and only if $\mathcal{A}$ is wild.

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  for some $1 \leq i \leq n(d)$ and some $\lambda \in k$.

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Verified in some special cases: Finite representation type, tame hereditary algebras, tame concealed algebra, tubular algebras.

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PFD: Wild $\Rightarrow$ Undecidable

($k = \overline{k}$) Same good partial results as for Prest's conjecture.

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Definition

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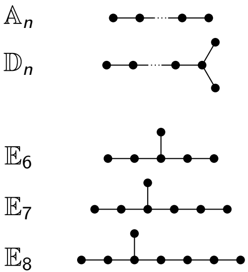
where

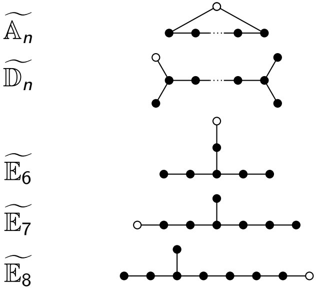
- for each $i \in Q_0$, V_i is a k-vector space and
- for each $i \xrightarrow{\alpha} j \in Q_1$, $\Phi_\alpha : V_i \to V_j$ is a k-linear map.

Tame Quivers

Dynkin Graphs



Extended Dynkin Graphs



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Then $Th(\text{fin-}\mathcal{A})$ is decidable.

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A (right) **pp-n-formula** (over R) is a formula $\varphi(\overline{x})$ of the form

$$\exists y_1, \dots, y_m \bigwedge_{i=1}^{l} \sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} = 0$$

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Example: The indecomposable pure-injective abelian groups are:

- For each $p \in \mathbb{P}$ and $i \in \mathbb{N}$, $\mathbb{Z}/p^i\mathbb{Z}$.
- The Prüfer group $\mathbb{Z}_{p^\infty}$.
- The p-adic group $\widehat{\mathbb{Z}_p}$.
- $\mathbb{Q}$

Pure-Injective Abelian Groups

$\mathbb{Q}$

$\varprojlim \mathbb{Z}/2^i\mathbb{Z} =: \widehat{E_2}$ $\quad E_2[\infty] =: \varinjlim \mathbb{Z}/2^i\mathbb{Z}$

$\varprojlim \mathbb{Z}/p^i\mathbb{Z} =: \widehat{E_p}$ $\quad E_p[\infty] =: \varinjlim \mathbb{Z}/p^i\mathbb{Z}$

p-adic numbers

p-Prüfer group

$\mathbb{Z}/8\mathbb{Z} =: E_2[3]$

$\mathbb{Z}/p^3\mathbb{Z} =: E_p[3]$

$a + 4\mathbb{Z}$ $\quad \mathbb{Z}/4\mathbb{Z} =: E_2[2]$ $\quad a \cdot 2 + 4\mathbb{Z}$

$a + p^2\mathbb{Z}$ $\quad \mathbb{Z}/p^2\mathbb{Z} =: E_p[2]$ $\quad ap + p^2\mathbb{Z}$

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$p \in \mathbb{P}$

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Theorem (Basarab; Herzog & Rothmaler)

For $M \in \text{Mod-}\mathbb{Z}$ the following conditions are equivalent.

- *M is pseudofinite.*

Pseudofinite Abelian Groups

An abelian group is **pseudofinite** if it satisfies all sentences in $\mathrm{Th}(\text{fin-}\mathbb{Z})$.

Fact: There is an order anti-isomorphism $D : \mathrm{pp}^1_R \to {}_R\mathrm{pp}^1$.

Theorem (Basarab; Herzog & Rothmaler)

For $M \in \text{Mod-}\mathbb{Z}$ the following conditions are equivalent.

- *M is pseudofinite.*
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Example: Let $p \in \mathbb{P}$. For all $M \in \text{Mod-}\mathbb{Z}$,

$$|{}^{xp=0}/{}_{x=0}(M)| = |\mathrm{ann}_M p| = |\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, M)| \text{ and}$$

$$|{}^{D(x=0)}/{}_{D(xp=0)}(M)| = |{}^{x=x}/{}_{p|x}(M)| = |M/Mp| = |\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, M)|.$$

Pure-Injective Abelian Groups

$\mathbb{Q}$

$\varprojlim \mathbb{Z}/2^i\mathbb{Z} =: \widehat{E_2}$ $\quad E_2[\infty] =: \varinjlim \mathbb{Z}/2^i\mathbb{Z}$

$\varprojlim \mathbb{Z}/p^i\mathbb{Z} =: \widehat{E_p}$ $\quad E_p[\infty] =: \varinjlim \mathbb{Z}/p^i\mathbb{Z}$

p-adic numbers

p-Prüfer group

$\mathbb{Z}/8\mathbb{Z} =: E_2[3]$ $\qquad \mathbb{Z}/p^3\mathbb{Z} =: E_p[3]$

$a + 4\mathbb{Z}$ $\quad \mathbb{Z}/4\mathbb{Z} =: E_2[2]$ $\quad a \cdot 2 + 4\mathbb{Z}$ $\qquad a + p^2\mathbb{Z}$ $\quad \mathbb{Z}/p^2\mathbb{Z} =: E_p[2]$ $\quad ap + p^2\mathbb{Z}$

$a + 2\mathbb{Z}$ $\quad \mathbb{Z}/2\mathbb{Z} =: E_2[1]$ $\quad a + 2\mathbb{Z}$ $\qquad a + p\mathbb{Z}$ $\quad \mathbb{Z}/p\mathbb{Z} =: E_p[1]$ $\quad a + p\mathbb{Z}$

$p \in \mathbb{P}$

Pure - Injective ~~Abelian Groups~~ k[t] - modules

$k(t)$

$\varprojlim E_{\langle t\rangle}[i] =: E_{\langle t\rangle}$ $\quad E_{\langle t\rangle}[\infty] := \varinjlim E_{\langle t\rangle}[i]$

$k[t]/\langle t\rangle^3 =: E_{\langle t\rangle}[3]$

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$\mathfrak{p} \lhd k[x]$ prime

Pseudofinite-dimensional $k[t]$-modules

Pseudofinite-dimensional $k[t]$-modules

Theorem

For $M \in \text{Mod-}k[t]$ the following conditions are equivalent.

- *M is pseudofinite-dimensional.*
- *For every pair of pp-formulae φ/ψ and $m \in \mathbb{N}$,*

  $$|\varphi/\psi(M)| \geq m \text{ if and only if } |D\psi/D\varphi(M)| \geq m.$$

- *M is elementary equivalent to a direct sum of finite-dimensional $k[t]$-modules, $E_{\mathfrak{p}}[\infty] \oplus \widehat{E_{\mathfrak{p}}}$ for some prime $\mathfrak{p} \lhd k[t]$ and $k(t)$.*

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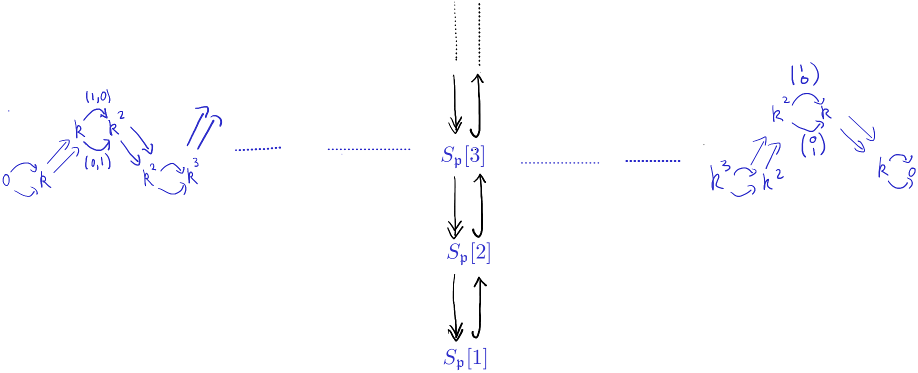
Pure-Injective kK_2-modules

$S_\mathfrak{p}[i] := E_\mathfrak{p}[i] \overset{\cdot t}{\underset{1}{\rightrightarrows}} E_\mathfrak{p}[i]$

$S_\mathfrak{p}[\infty] := E_\mathfrak{p}[\infty] \overset{\cdot t}{\underset{\cdot 1}{\rightrightarrows}} E_\mathfrak{p}[\infty]$

$\mathcal{G} := k(t) \overset{\cdot t}{\underset{\cdot 1}{\rightrightarrows}} k(t)$

$\widehat{E_\mathfrak{p}} \overset{\cdot t}{\underset{\cdot 1}{\rightrightarrows}} \widehat{E_\mathfrak{p}} =: \widehat{S_\mathfrak{p}}$



$\mathfrak{p} \in \{\mathfrak{p} \triangleleft k[t] \mid \mathfrak{p} \text{ prime}\} \cup \{\infty\}$

Pseudofinite-dimensional $k\mathbb{K}_2$-modules

$$\mathbb{K}_2 := \; 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

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The indecomposable pure-injective $k\mathbb{K}_2$-modules are the finite-dimensional indecomposable $k\mathbb{K}_2$-modules, $S_{\mathfrak{p}}[\infty]$, $\widehat{S_{\mathfrak{p}}}$ and $\mathcal{G}$.

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Theorem (G.)

The pseudofinite $k\mathbb{K}_2$-modules are those elementary equivalent to a direct sum of finite-dimensional modules,

$$\widehat{S_{\mathfrak{p}}} \oplus S_{\mathfrak{p}}[\infty], \quad \mathcal{G}, \quad \bigoplus_{\mathfrak{p}\in\mathbb{P}\cup\{\infty\}} \widehat{S_{\mathfrak{p}}} \quad \textit{and} \quad \bigoplus_{\mathfrak{p}\in\mathbb{P}\cup\{\infty\}} S_{\mathfrak{p}}[\infty].$$

–The End–

Ingredients of the axiomatisation

Fact Let $\mathcal{A}$ be tame hereditary. For all $M, N \in \text{fin-}\mathcal{A}$, the value of

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Fact For any $X \in \text{fin-}\mathcal{A}$, there are pairs of pp-formulae φ/ψ and σ/τ such that for all $M \in \text{Mod-}\mathcal{A}$

$$|\operatorname{Hom}(X, M)| = |{}^{\varphi}/_{\psi}(M)| \text{ and } |\operatorname{Ext}(X, M)| = |{}^{\sigma}/_{\tau}(M)|.$$

Fact

Let $\mathcal{A}$ be a finite-dimensional algebra. Let φ/ψ be a pp-pair and let $\mathcal{K}$ be a finite set of indecomposable finite-dimensional $\mathcal{A}$-modules. There is a pp-pair $[\varphi/\psi]_{\mathcal{K}}$ such that for all $K \in \mathcal{K}$,

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Let $\mathcal{A}$ be a tame hereditary algebra over an infinite field. An $\mathcal{A}$-module is pseudofinite-dimensional if and only if it satisfies the following sentences.

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$$|[\text{Ext}(X, -)]_{\mathcal{K}}| = 1 \vee |[\text{Ext}(Y, -)]_{\mathcal{K}}| > 1 \vee |[\text{Hom}(X, -)]_{\mathcal{K}}| > 1$$

–Thank you–