Quasiminimality and exponential algebraic closedness

Jonathan Kirby

University of East Anglia
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Abstract

Quasiminimality and exponential algebraic closedness

In this series of talks I will survey some of the work done towards understanding the model theory of exponential fields, including the real exponential and Zilber's approach to the complex exponential field. We know from Wilkie that the real exponential field is not too complicated (it is o-minimal) and this has good consequences in geometry, in number theory, and even in machine learning. For the complex exponential, we do not know if it is tame (quasiminimal) or whether it is maximally complicated (interpreting both reals and integers). I will explain progress towards proving that it is tame.

1. The o-minimal approach and the pregeometry of exponential algebraic closure
2. The algebra of exponential fields: kernels and strong extensions
3. Zilber's exponential fields and the conjectures
4. More about quasiminimality

Outline

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Vahagn Aslanyan, Jonathan Kirby, and Vincenzo Mantova.
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Gareth Boxall.
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Exponential fields

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \langle F; +\rangle \to \langle F \setminus \{0\}; \times\rangle$.

Examples

$\mathbb{R}_{\exp}$, with kernel $\{0\}$, and $\mathbb{C}_{\exp}$, with kernel $2\pi i\mathbb{Z}$.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We can also construct exponential maps "by hand":

1. Fix a field F, char 0.
2. Choose a $\mathbb{Q}$-linear basis $\{b_i\}_{i\in I}$ of F.
3. For each i, choose a non-zero $c_i \in F$.
4. Define $\exp(b_i) = c_i$, and extend $\mathbb{Q}$-linearly to an exponential map.

There are still choices to be made like $\exp(b_i/2) = \pm\sqrt{c_i}$, but it can be done, at least if F has enough roots.

Complex field

- Axioms: ACF_0
- strongly minimal
- algebraic closure
- models = alg closed sets
- uncountable categoricity
- good structure theory of definable sets: algebraic varieties, Zariski topology, dimension theory

Real Field

- Axioms: RCF
- o-mininal
- model-theoretic acl = dcl = field-theoretic (relative) algebraic closure,
- models = alg closed sets
- unstable, so no structure theory for models
- good structure theory for definable sets (cell-decomposition, dimension theory) and topology (Euclidean, definable sets have no bad singularities)

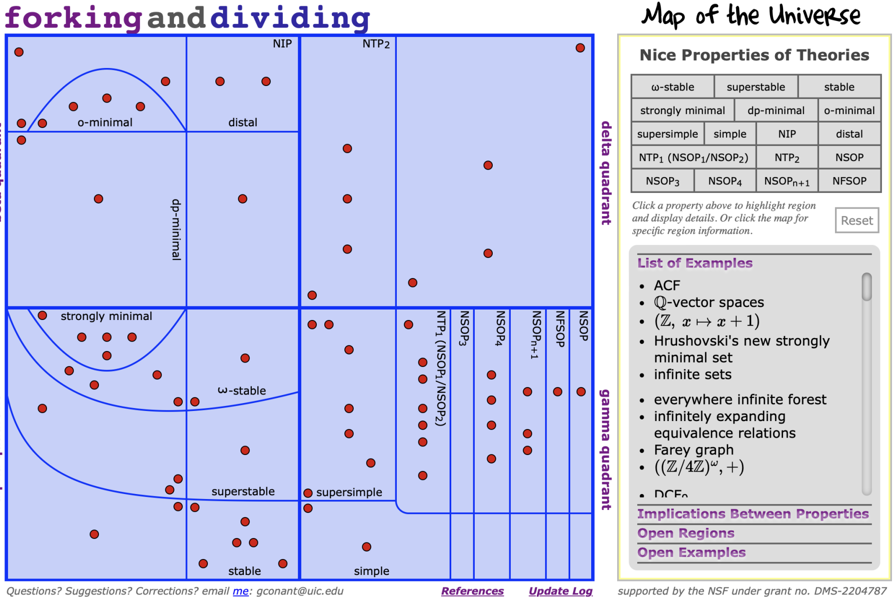
Complex exponential field

- Undecidable first-order theory: $\mathbb{Z}$ definable
- First-order theory in wildest part of stability hierarchy
- Open question: is $\mathbb{R}$ definable?
- Zilber conjecture: if not, maybe all is not lost

Real exponential field

- Decidability unknown (yes if Schanuel Conjecture true - Macintyre/Wilkie)
- o-minimal (Wilkie)
- model-theoretic acl = dcl = ??? (see later)

From Gabriel Conant's Forking and Dividing website



O-minimality

Definition

A expansion $\langle M; <, \cdots \rangle$ of DLO is o-minimal if every parameter-definable subset of M is a finite union of points and open intervals.

Fact (van den Dries book, p21)

Any o-minimal field is real-closed.

Lemma

Any o-minimal field has Skolem functions.

Corollary

dcl-closed sets (= subsets closed under $\emptyset$-definable functions) = elementary submodels

Question

What are the definable functions in $\mathbb{R}_{\exp}$?

Cell decomposition

Theorem (Weak, less technical version)

Any o-minimal structure M has cell decomposition: for all $n \in \mathbb{N}$, every parameter-definable subset of M^n is a finite union of cells.

Cells are definable sets which are in definable bijection to a Cartesian product of points and open intervals. (Not all such are cells.)

- *In an o-minimal field, for any $p \in \mathbb{N}$, one can take the definable bijections to be (definably) C^p (p-times continuously differentiable).*
- *In $\mathbb{R}_{\exp}$ (and in $\mathbb{R}_{\mathrm{an}}$ and its reducts), one can take the definable bijections to be real-analytic.*
- *In particular, any $\mathbb{R}_{\exp}$-definable function is piecewise real-analytic.*

Implicit functions are definable

Question

What are the definable functions in $\mathbb{R}_{\exp}$?

Polynomial functions and exponential polynomial functions are definable.

$$\text{E.g.} \qquad f(x) = x^2 + e^{e^{x^3} - 4e^x} + e^{2+x} + 4x.$$

The real logarithm is definable: $y = \log x$ iff $e^y = x$.

More generally, any implicit function of exponential polynomials is (locally) definable.

Implicit function theorem

See wikipedia! or some other reference

We could call an implicit function of a system of exponential polynomial functions an exponentially-algebraic function.

Implicit functions and dcl

We characterise the image points of implicitly defined functions. Let F be an o-minimal field.

Definition

$a_1 \in F$ is implicitly defined over a subset B in F iff for some $n \in \mathbb{N}$ there are:

- $\bar{a} = (a_1, \ldots, a_n) \in F^n$,
- A definable open subset $U \subseteq F^n$ containing $\bar{a}$,
- B-definable functions $f_1, \ldots, f_n : U \to F$

such that

- $f_i(\bar{a}) = 0$ for each $i = 1, \ldots, n$, and
- $\begin{vmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{vmatrix} (\bar{a}) \neq 0.$

Theorem

In $\mathbb{R}_{\exp}$ we have: $a \in \mathrm{dcl}(B)$ if and only if a is implicitly defined from B.
Furthermore, the functions f_i can be taken to be of the form $p_i(\bar{X}, \exp(\bar{X}))$ where p_i is a polynomial with coefficients in $\mathbb{Z} \cup B$.

The complex logarithm

Before looking at $\mathbb{C}_{\exp}$, let us consider the Complex logarithm.

Definition

The structure $\mathbb{C}_{\log} = \langle \mathbb{C}; +, \cdot, -, 0, 1, \mathrm{Log} \rangle$, *where* $\mathrm{Log} : \mathbb{C}^{\times} \to \mathbb{C}$ *is the branch of the complex logarithm such that* $-\pi < \operatorname{Im} \mathrm{Log}(z) \leqslant \pi$*.*

Exercise

$\mathbb{R}$ is definable in $\mathbb{C}_{\log}$.

Corollary

$\mathbb{C}_{\log}$ is interdefinable with $\mathbb{R}_{\exp, \sin\restriction_{[0,2\pi]}}$.

Hence we are really still in the o-minimal setting.
The complex exponential function is locally definable in $\mathbb{C}_{\log}$.

Holomorphic closure

In $\mathbb{C}_{\log}$, dcl is really a closure operator, and pregeometry, on $\mathbb{R}$, not on $\mathbb{C}$.
However, implicit closure, with $F = \mathbb{C}$, $U \subseteq \mathbb{C}^n$, and the extra condition that the functions $f_i : U \to \mathbb{C}$ are holomorphic, does give a pregeometry on $\mathbb{C}$, an example of holomorphic closure.
If we take account of complex conjugation (which is definable), we get back the definable closure on $\mathbb{R}_{\exp, \sin\restriction_{[0,2\pi]}}$ — every definable real function comes from a definable holomorphic function.

Other functions

Holomorphic closure is a sensible notion for reducts of $\mathbb{R}_{\mathrm{an}}$, especially expansions of $\mathbb{R}_{\mathrm{field}}$ by the real and imaginary parts of a set of holomorphic functions.

- It is a useful tool in understanding all the locally definable holomorphic functions in such a reduct.
- Wilkie made a conjecture about these. The conjecture was (refuted and) refined by Jones, Kirby, Le Gal, and Servi.
- In specific cases, such as for exponentiation and Weierstrass $\wp$-functions, we can prove things about these locally definable functions.
- Ax–Schanuel theorems of functional transcendence are important for this work.
- Raymond McCulloch has further work in this direction.

Pregeometries

Definition

Let X be a set. A pregeometry on X is an operation $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A, B \subseteq X$:

1. $A \subseteq \mathrm{cl}(A)$
2. $A \subseteq B \implies \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
3. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
4. *(Finite character) if $b \in \mathrm{cl}(A)$ then there is a finite subset $A_0 \subseteq A$ such that $b \in \mathrm{cl}(A_0)$*
5. *(Steinitz exchange) If $a \in \mathrm{cl}(A \cup \{b\}) \smallsetminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(A \cup \{a\})$.*

Closed sets have a cardinal dimension, like vector space dimension etc.

Given an E-field F, there are various useful pregeometries:

- $\mathbb{Q}$-linear span / $\mathbb{Q}$-linear dimension
- $\mathbb{Q}$-linear span on mult group / multiplicative rank
- field-theoretic algebraicity / transcendence degree
- exponential algebraicity / exponential transcendence degree

Exponential algebraic closure

Polynomials and exponential polynomials

a is algebraic over B iff there is a non-trivial polynomial $p(X, \bar{Y}) \in \mathbb{Q}[X, \bar{Y}]$ and $\bar{b} \in B$ such that $p(a, \bar{b}) = 0$.

$$a^5 + b_1 a^3 + b_2^7 a + b_3 = 0$$

Exponential polynomials do not capture exponential algebraicity alone:

$$e^{a^2} + b_1 e^{e^a} + e^{b_2} = 0$$

In $\mathbb{R}_{\exp}$, $a_1 \in \mathrm{dcl}(B)$ if there are n and $a_2, \ldots, a_n$ and n "different" exponential polynomials satisfied by $\bar{a}$ over B.
Formalised by the notion of an implicit system of exponential polynomials.
In $\mathbb{C}_{\exp}$, the same definition also gives a pregeometry, which coincides with holomorphic closure.

Example

- The zeros of $f(z) = e^z - z$ are the complex points a such that $e^a = a$.
- At such a point a, $f'(a) = e^a - 1 \neq 0$.
- There are infinitely many of them, and they are isolated, so there are countably many.
- In $\mathbb{R}_{\exp,\sin\restriction_{[0,2\pi]}}$, their real and imaginary parts are in $\mathrm{dcl}(\emptyset)$.
- In $\mathbb{C}_{\exp}$, they may not be in the model-theoretic algebraic closure of $\emptyset$.

Exponential Algebraicity

Let $\langle F; +, \cdot, \exp\rangle$ be any exponential field.

Definition

$a_1 \in F$ is exponentially algebraic over a subset B in F iff for some $n \in \mathbb{N}$ there are:

- $\bar{a} = (a_1, \ldots, a_n) \in F^n$
- polynomials $p_1, \ldots, p_n \in \mathbb{Z}[\bar{X}, e^{\bar{X}}, \bar{Y}]$
- $\bar{b}$ from B

such that setting $f_i(\bar{x}) = p_i(\bar{x}, e^{\bar{x}}, \bar{b})$ we have

- $f_i(\bar{a}) = 0$ for each $i = 1, \ldots, n$, and
- $\begin{vmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{vmatrix} (\bar{a}) \neq 0.$

Exponentially Transcendental over B in F $\iff$ not exponentially algebraic over B in F

Theorem

Exponential-algebraic closure, ecl, is a pregeometry on any exponential field.

Theorem (Countable Closure Property)

In $\mathbb{C}_{\exp}$ and $\mathbb{R}_{\exp}$, if B is countable then $\mathrm{ecl}(B)$ is also countable.

Towards Zilber's exponential field

Recall

- $\mathbb{C}$ is a field defined by analytic means (metric completion / Dedekind cuts / ...).
- Algebraic approach starts with $\mathbb{Q}$ and considers (finitely generated) field extensions.
- Amalgamate them (suitably complete colimit) to get notion of Algebraically Closed Field.
- Categoricity: up to isomorphism, there is a unique ACF (char 0, cardinality $2^{\aleph_0}$). Call it K.
- Fundamental Theorem of Algebra says that $\mathbb{C} \cong K$.

Goal:

- $\mathbb{C}_{\exp}$ is a E-field defined by analytic means.
- Algebraic approach starts where? (SK)
- What are the finitely generated / finitely presented extensions?
- Which can we amalgamate? (strong extensions)
- Get notion of Exponentially-Algebraically Closed Field.
- Categoricity: up to isomorphism, there is a unique Exp-Alg-Closed Field (with Schanuel Property, Standard Kernel, CCP, cardinality $2^{\aleph_0}$). Call it $\mathbb{B}$.
- Is $\mathbb{C}_{\exp}$ Exponentially-Algebraically Closed?
- Is $\mathbb{C}_{\exp} \cong \mathbb{B}$?

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \langle F; +\rangle \to \langle F \setminus \{0\}; \times\rangle$.
If the field is algebraically closed we call it an EA-field.
If additionally the exponential map is surjective we call it an ELA-field. (L for logarithm)
The kernel of F is $\{a \in F \mid \exp(a) = 1\}$.

Question

What subgroups can occur as the kernel of an exponential map?
What about for an ELA-field?

Example

In $\mathbb{C}_{\exp}$, the kernel is $\tau\mathbb{Z}$ for transcendental τ ($= 2\pi i$). We call this Standard Kernel.

Observe for F alg closed

$\langle F; +\rangle$ is a divisible, torsion-free abelian group.
$\langle F \setminus \{0\}; \times\rangle$ is divisible, abelian, and the torsion is $\sqrt{1}$, the roots of unity.
For each $n \in \mathbb{N}^+$, the subgroup of n-torsion, $\{a \mid a^n = 1\}$, has size n.

Full kernel

Theorem

Suppose that F is an E-field, with all roots of unity in F. Let K be the kernel. TFAE:

1. All roots of unity are in the image of $\exp$
2. $\mathbb{Q}K/K \cong \sqrt{1}$, the multiplicative group of all roots of unity
3. For each $n \in \mathbb{N}^+$, K/nK is a cyclic group of order n
4. For each $n \in \mathbb{N}^+$, $|K/nK| = n$
5. The profinite completion $\hat{K}$ of K satisfies $\hat{K} \cong \hat{\mathbb{Z}}$

Any subgroup K satisfying these properties can be the kernel of an exponential map.

Note that condition 4 is first-order axiomatizable in the language of groups.

Fact (part of Szmielew's theorem, see Hodges 1993, A.2.7)

The complete first order-theory of an abelian group is determined by a certain list of invariants. In the case of $\langle \mathbb{Z}; + \rangle$, torsion-free + 4 above is enough to determine the complete theory.

So, we can add to the above list:

6. $\langle K; + \rangle$ is elementarily equivalent to $\langle \mathbb{Z}; + \rangle$.

Fact

Any model of $\mathrm{Th}(\mathbb{Z}; +)$ is of the form $D \oplus R$ where D is divisible torsion-free (so a $\mathbb{Q}$-vector space), and R is isomorphic to a subgroup of $\hat{\mathbb{Z}}$. The theory is superstable.

Constructions of exponential fields

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \langle F; +\rangle \to \langle F \setminus \{0\}; \times\rangle$.
If the field is algebraically closed we call it an EA-field.
If additionally the exponential map is surjective we call it an ELA-field. (L for logarithm)

Apart from the analytic examples $\mathbb{R}_{\mathrm{exp}}$, $\mathbb{C}_{\mathrm{exp}}$, we can also construct exponential maps "by hand":

1. Fix a field F, char 0.
2. Choose a $\mathbb{Q}$-linear basis $\{b_i\}_{i\in I}$ of F.
3. For each i, choose a non-zero $c_i \in F$.
4. Define $\exp(b_i) = c_i$, and extend $\mathbb{Q}$-linearly to an exponential map.

There are still choices to be made like $\exp(b_i/2) = \pm\sqrt{c_i}$, but it can be done, at least if F has enough roots.

Free extensions

Free extension construction

Fix an E-field F with "full kernel" – all roots of unity in image of $\exp$.
In fact, suppose the exponential map is defined only on a $\mathbb{Q}$-linear subspace D of F.

- Choose a $\mathbb{Q}$-linear basis $(b_i)_{i \in I}$ of F over D.
- Take $(c_{i,1})_{i \in I}$, algebraically independent over F.
- For $n \in \mathbb{N}^+$, now take $c_{i,n}$ such that for all $r, m \in \mathbb{N}^+$ we have $c_{i,rm}^r = c_{i,m}$.
  Coherent system of roots of $c_{i,1}$.
- For $i \in I, m \in \mathbb{N}^+$, define $\exp(b_i/m) = c_{i,m}$ and extend by additivity to an exponential map.

We have constructed a new field F^e with an exponential map defined on $D(F^e) = F$.
Now iterate:

$$F \hookrightarrow F^e \hookrightarrow F^{ee} \hookrightarrow \cdots$$

to get at stage ω an E-field F^E, the free E-field extension.

Example

If F is already an E-field, then $F(a)^E$ is the free-E-field extension on one generator, a.

Free extensions 2

Free extension construction

We constructed a new field F^e with an exponential map defined on $D(F^e) = F$.
Now iterate:

$$F \hookrightarrow F^e \hookrightarrow F^{ee} \hookrightarrow \cdots$$

to get at stage ω an E-field F^E, the free E-field extension.

Variant

Write F^a for the algebraic closure of F, with the same (partially defined) exp map as F.
Iterating

$$F \hookrightarrow F^a \hookrightarrow F^{ae} \hookrightarrow F^{aea} \hookrightarrow F^{aeae} \hookrightarrow \cdots$$

we get F^{EA}, the free EA-field extension of F.

Lemma

Given F, a partial E-field with full kernel, the free extensions F^E and F^{EA} are unique up to isomorphism over F.

Proof idea.

The only choices apparently made are to choose the coherent systems of roots of the c_i. But the c_i are algebraically independent over the field F and there is only one type of a coherent system. □

Free extensions 3

Now we want to make our exponential map surjective.

Free logarithm construction

Fix an E-field F with "full kernel" – all roots of unity in image of $\exp$.
In fact, suppose the exponential map is defined only on a $\mathbb{Q}$-linear subspace D of F, and the image is $I(F)$.

- Choose a multiplicative basis $(b_i)_{i \in I}$ of $F^\times$ over $I(F)$.
- For $n \in \mathbb{N}^+$, now choose $b_{i,n}$ such that for all $r, m \in \mathbb{N}^+$ we have $b_{i,rm}^r = b_{i,m}$. These may be in F or in F^a.
- Take $(a_i)_{i \in I}$, algebraically independent over F.
- Let F^l be the field $F(a_i, b_{i,m})_{i \in I, m \in \mathbb{N}^+}$, with:
- For $i \in I, m \in \mathbb{N}^+$, define $\exp(a_i/m) = b_{i,m}$ and extend by additivity to an exponential map.

We have constructed a new field F^l with an exponential map whose image contains F.
Now iterate:

$$F \hookrightarrow F^e \hookrightarrow F^{el} \hookrightarrow F^{ela} \hookrightarrow F^{elae} \hookrightarrow F^{elael} \hookrightarrow F^{elaela} \hookrightarrow \cdots$$

to get at stage ω an E-field F^{ELA}, the free ELA-field extension.

There are choices to be made of the coherent systems of roots in the l stages.

Uniqueness of free extensions

There are choices to be made of the coherent systems of roots in the I stages.
However, we still get uniqueness in important cases.

Theorem (Zilber's Thumbtack Lemma / Open Image theorem, versions 1,2)

Let $F = K(a_1, \ldots, a_s, \sqrt{b_1}, \ldots, \sqrt{b_r})$, where K is $\mathbb{Q}(\sqrt{1})$ or an algebraically closed field of characteristic zero. Suppose that c lies in some field extension of F and is multiplicatively independent from $K^\times \cdot \langle b_1, \ldots, b_r \rangle$. Then there is $m \in \mathbb{N}$ and an m root c_m of c such that there is exactly one isomorphism type of a coherent system of roots of c_m over F.

Corollary

*Suppose F is a partial E-field with full kernel such that the domain D of exponentiation is either a finite-dimensional $\mathbb{Q}$-vector space, or is finite dimensional over a countable ELA-field F_0.
Then F^{ELA} is uniquely defined up to isomorphism as an extension of F.*

Proof ideas

- We choose the coherent systems of roots one at a time, each time replacing c by $\sqrt[m]{c}$ for an appropriate m, and applying the theorem to get uniqueness.
- We use countability of F^{ELA} to ensure that we can always work over an extension of an algebraically closed K which is generated by a finite set together with a finite set of coherent systems of roots, so the theorem remains applicable. □

A similar method gives a unique smallest ELA-field $\mathrm{SK}^{\mathrm{ELA}}$ with kernel $\tau\mathbb{Z}$, τ transcendental.

Finitely presented extensions

Setup

Let $F \subseteq M$ be an extension of ELA-fields. Suppose $a = (a_1, \ldots, a_n)$ is in M, and F_1 is the partial E-field extension of F with domain given by the $\mathbb{Q}$-linear span of $F \cup \{a_1, \ldots, a_n\}$.

Question

What information determines the isomorphism type of F_1?

Answer

- We can assume the generating set is $\mathbb{Q}$-linearly independent over F.
- Assume also that there are no new kernel elements, so $e^{a_1} \ldots, e^{a_n}$ are multiplicatively independent over F.
- For $m \in \mathbb{N}^+$, let $V_m = \mathrm{Loc}(a_1/m, \ldots, a_n/m, e^{a_1/m}, \ldots, e^{a_n/m}/F)$.
- It is clear that V_{rm} determines V_m for $r, m \in \mathbb{N}^+$.
- Via Thumbtack Lemma version 3, there is $N \in \mathbb{N}^+$ such that V_N determines all the V_m.
- This V_N is an algebraic variety, so is given by a finite list of polynomials.

Definition

We say the extension $F \subseteq F_1$ is a finitely presented extension of partial E-fields.
For countable F, we also say that $F \subseteq F_1^{ELA}$ is a finitely presented extension of ELA-fields.
For $V = V_N$ above, we write F_1^{ELA} as $F|V$.

Question: What is V for a free extension on n generators?

Schanuel's conjecture

Some transcendence statements

1. e is transcendental (conjectured by Lambert 1768, proved by Hermite 1873).
2. π is transcendental (conjectured by Lambert 1768, proved by Lindemann 1882).
3. If a, b are algebraic, $a \neq 0, 1$, b irrational, then a^b is transcendental. (Gelfond-Schneider theorem, 1934)
4. So $e^\pi = (e^{-i\pi})^i = (-1)^i$ is transcendental.
5. Baker's theorem (1966,1967) is a generalisation of Gelfond-Schneider.

Algebraic independence statements

1. e, e^π are algebraically independent (Nesterenko 1996)
2. Are e, π algebraically independent?

Schanuel's Conjecture

Suppose $a_1, \ldots, a_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent. Then

$$\mathrm{td}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \geqslant n.$$

Predimension

Schanuel's Conjecture

Suppose $a_1, \ldots, a_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent. Then

$$\mathrm{td}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \geqslant n.$$

Definition

Given $a = (a_1, \ldots, a_n)$ in an E-field, define

$$\delta(a) := \mathrm{td}(a, e^a) - \mathrm{ldim}_{\mathbb{Q}}(a).$$

Then Schanuel's conjecture is equivalent to: for all $n \in \mathbb{N}$, for all $a \in \mathbb{C}^n$, $\delta(a) \geqslant 0$.

Relative predimension

Let F be an E-field, $B \subseteq F$ and $a \in F^n$.

$$\delta(a/B) := \mathrm{td}(a, e^a/B, \exp(B)) - \mathrm{ldim}_{\mathbb{Q}}(a/B)$$

The relative predimension satisfies the additive property:

$$\delta(ab/C) = \delta(a/bC) + \delta(b/C)$$

Proof.

Both td and $\mathrm{ldim}_{\mathbb{Q}}$ satisfy the additive property. □

Goal

We want to construct an exponential field satisfying the Schanuel property (for all a, $\delta(a) \geqslant 0$).
We build it up via an amalgamation process, amalgamating finitely generated ELA-subfields.
We start with SK, the standard kernel.
Previously we characterised finitely presented extensions. Now we need to see which of these satisfy the Schanuel property, and how to preserve it.

Free extensions are strong

Definition

An extension $F \subseteq F_1$ of E-fields (or partial E-fields) is strong, written $F \lhd F_1$, if for all tuples $a \in F_1$ we have $\delta(a/D(F)) \geqslant 0$.

Lemma (Free extensions are strong)

For any partial E-field F, we have $F \lhd F^E$, $F \lhd F^{EA}$ and $F \lhd F^{ELA}$.

Corollary

SK^{ELA} satisfies the Schanuel property.

Strong extensions are (often) free

Theorem

Suppose M is an ELA-field, that $F \lhd M$ is a strong partial E-subfield, and that K is the ELA-closure of F in M. Then $K \cong F^{ELA}$.

SK embeds in $\mathbb{C}_{\exp}$. Schanuel's conjecture is equivalent to $SK \lhd \mathbb{C}_{\exp}$.

Definition (Ritt 1948?, Chow 1999)

The ELA-field $\mathbb{L}$ of Liouvillian numbers is the smallest ELA-subfield of $\mathbb{C}_{\exp}$. Its smallest EL-subfield $\mathbb{E}$ consists of all complex numbers with a closed form representation in terms of $\exp$ and the principal logarithm Log.

Corollary

If Schanuel's conjecture is true, then $\mathbb{L} \cong SK^{ELA}$. Furthermore, there is an algorithm for deciding transcendence questions in $\mathbb{L}$, and hence also in $\mathbb{E}$.

Finitely generated ELA-field extensions – recap

Fix an ELA-field, F. Consider a partial E-field extension $F \subseteq F_1$ generated by $\bar{a} = (a_1 \dots, a_n) \in D(F_1)$. Let $V = \mathrm{Loc}(\bar{a}, \exp^{\bar{a}} / F)$, the algebraic locus.

- We may assume $\bar{a}$ is $\mathbb{Q}$-linearly independent over F. We say V is additively free.
- We assume the extension does not extend the kernel. Equivalently, $\exp(\bar{a})$ is multiplicatively independent over F. Say V is multiplicatively free.
- (Application of thumbtack lemma) Replacing $\bar{a}$ by $\bar{a}/N$ for some $N \in \mathbb{N}$ if necessary, F_1 is determined up to isomorphism as an extension of F by V.
- $F \subseteq F_1^{ELA}$ is a finitely generated ELA-field extension, also well-defined (at least when F is countable)
- Write $F_1^{ELA} = F|V$ where $V = \mathrm{Loc}(\bar{a}, \exp(\bar{a})/F)$ for generators $\bar{a}$ as before.

Definition

Suppose $F \subseteq K$ is a finitely generated extension of ELA-fields. If there is F_1 a finitely generated partial E-field extension of F such that $K \cong_F F_1^{ELA}$, we say that K is a finitely presented ELA-extension of F. We say that an appropriate variety V is the presentation.

So finitely generated strong extensions are very close to free extensions.

Theorem

If $F \lhd K$ is a finitely generated kernel-preserving strong extension of (countable) ELA-fields, then it is finitely presented.

Corollary

If Schanuel's conjecture is true then every finitely generated ELA-subfield of $\mathbb{C}_{\exp}$ is finitely presented. (We could say that $\mathbb{C}_{\exp}$ is locally finitely presented.)

Rotund varieties

Question

Given a finitely presented extension $F \subseteq F|V$ of ELA-fields, for which V is the extension strong?

We can assume V is additively and multiplicatively free. Then $F \lhd F|V$ if and only if V is rotund.

Furthermore, the extension $F \lhd F|V$ is exponentially algebraic if $\dim V = n$.
Otherwise exponentially transcendental, for example $n = 1$, $V = F \times F^{\times}$.

Recap

Standard kernel

Define a partial exponential field SK:

- Underlying field is $\mathbb{Q}(\sqrt{1}, \tau)$, with τ transcendental.
- Domain of exponential maps: $D(SK) = \mathbb{Q}\tau$
- $\exp(\tau/m)$ is a primitive m^{th} root of 1

This determines SK up to isomorphism.

SK embeds into $\mathbb{C}_{\exp}$ via

$$\tau \mapsto \pm 2\pi i$$

SK^{ELA} is the free ELA-completion of SK, unique up to isomorphism.
If Schanuel's conjecture is true, SK^{ELA} embeds into $\mathbb{C}_{\exp}$, and image is determined uniquely setwise, but SK^{ELA} has many automorphisms.

Strong extensions

Let F be an E-field, $B \subseteq F$ and $a \in F^n$.

$$\delta(a/B) := \mathrm{td}(a, e^a/B, \exp(B)) - \mathrm{ldim}_{\mathbb{Q}}(a/B)$$

Definition

An extension $F \subseteq F_1$ of E-fields (or partial E-fields) is strong, written $F \lhd F_1$, if for all tuples $a \in F_1$ we have $\delta(a/D(F)) \geqslant 0$, and the kernel does not extend.

Strong ELA-extensions

F an ELA-field. Given a suitable algebraic subvariety V of $F^n \times (F^\times)^n$, we can define an extension ELA-field $F_1 = F|V$ by adding a point $(a_1, \ldots, a_n) \in F_1^n$ such that $(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \in V$, and taking the free ELA-completion.

Suitable V means free and rotund: suppose $(\bar{x}, \bar{y})$ is generic in V over A.

Free : $\bar{x}$ is $\mathbb{Q}$-linearly independent over A and $\bar{y}$ is multiplicatively independent over A. So $\exp$ is well-defined and has no new kernel elements.

Rotund : $\dim V \geqslant n$, and similarly for $\mathbb{Q}$-linear projections. This ensures the extension is strong.

Proposition (Finite presentation for finitely generated strong extensions)

Finitely generated strong ELA-field extensions of ELA-fields F are all of the form $F \lhd F|V$, so for countable F are determined up to isomorphism by a single algebraic variety, V, hence by a finite amount of data.

Corollary

There are only countably many isomorphism types of finitely generated ELA-fields (with standard kernel), and only countably many isomorphism types of finitely generated strong extensions of a countable ELA-field.

Amalgamation

Theorem (A version of Fraïssé's Amalgamation theorem)

Suppose $\mathcal{C}$ is a category of countable structures and embeddings satisfying:

- *There are only countably many finitely generated objects in $\mathcal{C}$, up to isomorphism.*
- *For each finitely generated $A \in \mathcal{C}$, there are only countably many isomorphism classes of finitely generated extensions $A \hookrightarrow B$ in $\mathcal{C}$.*
- *$\mathcal{C}$ has the amalgamation property.*
- *$\mathcal{C}$ has the joint embedding property.*
- *$\mathcal{C}$ has unions of chains of length ω.*
- *Each $A \in \mathcal{C}$ is a union of a chain of finitely-generated objects in $\mathcal{C}$.*

Then there is a structure U in $\mathcal{C}$ which is universal in $\mathcal{C}$ (everything in $\mathcal{C}$ embeds in it), homogeneous and saturated (with respect to arrows in $\mathcal{C}$). Furthermore, U is unique up to isomorphism.

Take $\mathcal{C}$ the category of countable strong ELA-field extensions of SK^{ELA}.

- The finitely generated strong ELA-extensions are $F \lhd F|V$ for free and rotund V.
- There are only countably many such V, so only countably many such extensions.
- One can easily show $(F|V)|W \cong F|(V \times W) \cong (F|W)|V$, so we have amalgamation.
- SK^{ELA} embeds in everything so AP gives JEP.

The Fraïssé limit U is $\mathbb{B}_\omega$, the countable version of Zilber's exponential field.

Uncountable models

We have $\mathbb{B}_\omega$, a countable ELA-field with the Schanuel property and standard kernel which is universal and saturated for all such ELA-fields (with respect to strong extensions).
The exponential transcendence degree of $\mathbb{B}_\omega$ is $\aleph_0$.

Shelah / Zilber quasiminimal excellence method produces for each infinite κ, a unique model $\mathbb{B}_\kappa$ of cardinality and exponential transcendence degree κ. We define $\mathbb{B} = \mathbb{B}_{2^{\aleph_0}}$.

Without excellence:

Choose an exponential transcendence base $(b_i)_{i<\omega}$. Let $B' = \mathrm{ecl}_{\mathbb{B}_\omega}((b_i)_{0<i<\omega})$. Then B' is a proper ELA-subfield of $\mathbb{B}_\omega$ and $\mathrm{etd}(\mathbb{B}_\omega/B') = 1$.
However, $B' \cong \mathbb{B}_\omega$, and we have an inclusion map $B' \hookrightarrow \mathbb{B}_\omega$, which gives a self-map $f : \mathbb{B}_\omega \hookrightarrow \mathbb{B}_\omega$.
We relabel the copies of $\mathbb{B}_\omega$ to write this as $\mathbb{B}_\omega \hookrightarrow \mathbb{B}_{\omega+1}$.
Then iterate this map to get

$$\mathbb{B}_\omega \hookrightarrow \mathbb{B}_{\omega+1} \hookrightarrow \mathbb{B}_{\omega+2} \hookrightarrow \mathbb{B}_{\omega+3} \hookrightarrow \cdots \hookrightarrow \mathbb{B}_\alpha \hookrightarrow \cdots$$

where α ranges over countable ordinals.
At each stage we have a copy of $\mathbb{B}_\omega$ but with an exponential transcendence base $(b_\gamma)_{\gamma<\alpha}$, and inclusion maps which preserve the b_γ.
Let $\mathbb{B}_{\omega_1}$ be the union of the chain.
Each $\mathbb{B}_\alpha$ is ecl-closed in $\mathbb{B}_{\omega_1}$ and so $\mathbb{B}_{\omega_1}$ has the countable closure property.
The cardinality of $\mathbb{B}_{\omega_1}$ is $\aleph_1$. Now assume the Continuum Hypothesis, so $2^{\aleph_0} = \aleph_1$.

Axiomatic approach to Zilber's exponential field

Instead of doing amalgamation, Zilber gave a list of axioms and proved categoricity of them.

Axioms for Zilber's exponential fields

1. $\mathbb{B}$ is an ELA-field.
2. $\mathbb{B}$ has standard kernel.
3. The conclusion of Schanuel's conjecture holds (equivalently, $SK \lhd \mathbb{B}$).
4. $\mathbb{B}$ is strongly exponentially algebraically closed: for every free, rotund V of dimension n and every finite $\bar{b}$ there is $\bar{a}$ such that $(\bar{a}, \exp(\bar{a}))$ is generic in V over $\bar{b}$.
5. The countable closure property: For any countable $A \subseteq \mathbb{B}$ we have that $\mathrm{ecl}_{\mathbb{B}}(A)$ is countable. Equivalently, for each V as above, there are only countably many such $\bar{a}$ such that $(\bar{a}, \exp(\bar{a})) \in V$.

The axioms can be expressed as an $L_{\omega_1,\omega}(Q)$-sentence. We need $L_{\omega_1,\omega}$ just to omit the type of a non-standard kernel element, and then we need Q (there exist uncountably many ...) for the countable closure property.

Theorem

For each cardinal κ, there is exactly one model up to isomorphism of these axioms of exponential transcendence degree κ, and its cardinality is $\kappa + \aleph_0$.

Zilber's conjectures

Zilber's weak quasiminimality conjecture (1990s)

$\mathbb{C}_{\exp} = \langle \mathbb{C}; +, \cdot, \exp \rangle$ is quasiminimal: every definable subset of $\mathbb{C}$ is countable or co-countable.

Theorem

$\mathbb{B}$ is quasiminimal.
Indeed, given any countable $A \subseteq \mathbb{B}$, $\mathrm{ecl}_{\mathbb{B}}(A)$ is countable, and for any $a, b \in \mathbb{B} \smallsetminus \mathrm{ecl}_{\mathbb{B}}(A)$ there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{B}/\,\mathrm{ecl}_{\mathbb{B}}(A))$ such that $\sigma(a) = b$.
So every automorphism-invariant subset of $\mathbb{B}$ (over a countable set of parameters) is countable or co-countable.

Zilber's strong quasiminimality conjecture

$\mathbb{C}_{\exp} \cong \mathbb{B}$.

Remark

Since $\mathbb{B}$ is defined only up to isomorphism, the strong conjecture is really saying that $\mathbb{C}_{\exp}$ satisfies the axioms defining $\mathbb{B}$.

Theorem

Zilber's strong conjecture is equivalent to: Schanuel's conjecture and "$\mathbb{C}_{\exp}$ is strongly exponential-algebraically closed (SEAC)".

Exponential-algebraic closedness (EAC)

Corollary

If Schanuel's conjecture is true and $\mathbb{C}_{\exp}$ is strongly exponentially-algebraically closed (SEAC) then $\mathbb{C}_{\exp}$ is quasiminimal.

Definition (Strong Exponential-Algebraic Closedness)

F is SEAC if for every free and rotund subvariety V of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ of dimension n, and every finite tuple $\bar{b}$ in F, there is $\bar{a} \in F^n$ such that $(\bar{a}, e^{\bar{a}}) \in V$, generic in V over $\bar{b}$.

EAC / Zilber's Nullstellensatz

F is EAC if for every free and rotund subvariety V of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ of dimension n there is $\bar{a} \in F^n$ such that $(\bar{a}, e^{\bar{a}}) \in V$.

Conjecture (EAC conjecture)

Zilber's nullstellensatz holds for $\mathbb{C}_{\exp}$.

Theorem (Bays, K. 2018)

If $\mathbb{C}_{\exp}$ satisfies EAC then it is quasiminimal.

Progress towards EAC

Question

Given $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$, free and rotund, of dimension n, is there $a \in \mathbb{C}^n$ such that $(a, e^a) \in V$?

- $n = 1$ (Marker)
- Let $W = \mathrm{pr}_{\mathbb{C}^n} V$. If $\dim W = n$ then yes, (Masser, Brownawell, also D'Aquino, Fornasiero, Terzo).
- Same condition, geometric proof (Aslanyan, K, Mantova).
- Analogous case for exponential maps of Abelian varieties, (Aslanyan, K, Mantova).
- $\dim W = 1$ (Mantova, Masser)
- Exponential sums / complex powers (Gallinaro)
- Other abelian and j-examples (Gallinaro)
- j-situation with $\dim W = n$ (Eterovic, Herrera)
- Γ-function with $\dim W = n$ (Eterovic, Padgett)
- Some examples with j and its derivatives (Aslanyan, Eterovic, Mantova)

Question

For which complex functions f is $\langle \mathbb{C}; +, \cdot, f \rangle$ quasiminimal?

Examples

- complex conjugation – $\mathbb{R}$ definable so o-minimal, not quasiminimal
- j-function – domain is $\mathbb{H}$ or $\mathbb{H}^+ \cup \mathbb{H}^- = \mathbb{C} \setminus \mathbb{R}$

Generic functions (Dmitrieva in progress)

"Most" entire functions are quasiminimal.
In particular the Liouville functions defined by Wilkie and shown by him and Koiran to satisfy Zilber's first-order theory of a generic function
Also work of Le Gal on strongly transcendental functions

Examples

- Weierstrass $\wp$-functions – seem similar to $\exp$
- Exponential maps of abelian varieties $\exp_A : \mathbb{C}^g \to A(\mathbb{C})$. – seem similar again
- Fatou–Bieberbach example 1920s $f : \mathbb{C}^2 \to \mathbb{C}^2$ image open not dense – not QM.

Question

Koiran: Is the expansion of $\mathbb{C}$ by all 1-variable entire functions quasiminimal?
Equivalently, all 1-variable meromorphic on $\mathbb{C}$?

Some Quasiminimality theorems

Theorem (Bays, K 2018)

If $\mathbb{C}_{\exp}$ is exponentially algebraically closed (EAC) then it is quasiminimal.
Similar results for other expansions of $\mathbb{C}$ with known Ax–Schanuel theorem.

Using that technique:

Theorem (K 2019)

Let $\Gamma = \{(z, \exp(z + q + 2\pi i r)) \mid z \in \mathbb{C}, q, r \in \mathbb{Q}\}$.
Then the blurred exponential field $\langle \mathbb{C}; +, \cdot, \Gamma \rangle$ is quasiminimal.

Anna Dmitrieva has some similar examples around elliptic curves.

A variant of the technique, plus Gallinaro's proof of the relevant analogue of EAC gave:

Theorem (Gallinaro, K 2023)

For $\lambda \in \mathbb{C}$, let $\Gamma_\lambda = \{(\exp(z), \exp(\lambda z)) \mid z \in \mathbb{C}\}$, the graph of the multivalued map $w \mapsto w^\lambda$. Then the structure $\mathbb{C}$ with complex powers

$$\langle \mathbb{C}; +, \cdot, -, 0, 1, (\Gamma_\lambda)_{\lambda \in \mathbb{C}} \rangle$$

is quasiminimal.

Exponential sums equations

- In $\mathbb{C}_{\exp}$ we can express complicated equations like

$$e^{e^{\sin(z^2-iz)}} + \cos(e^{z-1/z}) + 1 = 0$$

In many applications, we do not iterate exponentiation but only use it to define complex powers: for fixed $\lambda \in \mathbb{C}$, define the multivalued $w^\lambda = \exp(\lambda \log w)$

- Similarly, consider exponential sums: for fixed $\lambda_i \in \mathbb{C}$, and $w_j = \exp(z_j)$, $\exp\left(\sum_{j=1}^n \lambda_j z_j\right) = \prod_{j=1}^n w_j^{\lambda_j}$.
- More generally for a matrix $M = (\lambda_{ij})$, let $\mathbf{u} = M\mathbf{z}$ and $v_i = \exp u_i$.

Definition

An exponential sums equation is an equation of the form $p(\mathbf{v}) = 0$, where $p \in \mathbb{C}[\mathbf{X}]$ and $\mathbf{v} = \exp(M\mathbf{z})$ as above.

A system of exponential sums equations gives an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ of the form $L \times W$ where $L \subseteq \mathbb{C}^n$ is given by $\mathbb{C}$-linear equations and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic subvariety. A solution is a point $\mathbf{a} \in L$ such that $\exp(\mathbf{a}) \in W$.

Theorem (Gallinaro, 2022, Nullstellensatz for complex exponential sums)

Suppose that $V = L \times W$ is a system of complex exponential sums equations which is free and rotund in the sense of exponential fields. Then there is a complex solution.

The complex powered field

Definition

Let $\mathbb{C}^{\mathbb{C}}$ be the complex numbers considered as a $\mathbb{C}$-powered field, that is:

$$\mathbb{C}_{\mathbb{C}\text{-VS}} \xrightarrow{\exp} \mathbb{C}_{\text{field}}$$

where the codomain is $\mathbb{C}$ equipped with the field structure, the cover is $\mathbb{C}$ equipped only with its structure as a $\mathbb{C}$-vector space, and the covering map is the usual complex exponentiation.

The equations expressible in this structure with variables in the cover are exactly the exponential sums equations.

The expressible equations with variables in the field are the "$\mathbb{C}$-powered polynomial" equations.

Theorem (Gallinaro, Kirby, 2023)

The complex powered field $\mathbb{C}^{\mathbb{C}}$ is quasiminimal.

So there should be some reasonable geometric theory of algebraic geometry with complex powers.

Almost all powers are generic

Theorem (Gallinaro, Kirby, 2023)

Let K be a countable field of characteristic 0. Then up to isomorphism, there is exactly one K-powered field $\mathbb{E}^K$ of cardinality continuum which:

(i) *has cyclic kernel,*

(ii) *satisfies the Schanuel property,*

(iii) *is K-powers closed, and*

(iv) *has the countable closure property.*

Furthermore, it is quasiminimal.

Theorem (Gallinaro, Kirby, 2023)

For all but countably many $\lambda \in \mathbb{C}$ (all exponentially transcendental λ), $\mathbb{C}^{\mathbb{Q}(\lambda)} \cong \mathbb{E}^{\mathbb{Q}(\lambda)}$.

If Schanuel's conjecture is true, the same holds for many other λ, including π, $2\pi i$.

Theorem (Zilber, unpublished)

The first-order theory of $\mathbb{E}^K$ is superstable.

Theorem (Gallinaro, 2022, Nullstellensatz for complex exponential sums)

$L \subseteq \mathbb{C}^n$, linear subspace, $W \subseteq (\mathbb{C}^\times)^n$, algebraic subvariety.
Suppose that $V = L \times W$ is a system of complex exponential sums equations which is free and rotund in the sense of exponential fields. Then there is a complex solution.

Definition

- Free: L is not contained in a proper $\mathbb{Q}$-linear subspace of $\mathbb{C}^n$, and W not contained in a coset of an algebraic subgroup of $(\mathbb{C}^\times)^n$.
- Rotund: $\dim L + \dim W \geqslant n$, and for any $M \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$, $\dim M \cdot L + \dim W^M \geqslant \mathrm{rk}\, M$.

Proof ideas

- Want to show $\mathbb{C}^n \supseteq L \cap \mathrm{Log}(W) \neq \emptyset$.
- Expect solutions to accumulate at infinity where $\exp$ has essential singularity.
- Use tropical geometry and amoebas to understand behaviour of W near 0 and ∞ — $\mathrm{Log}\, W$ becomes close to linear.
- For $K \subseteq \mathbb{R}$, now use results of Khovanskii and Zilber to get approximate solutions, and prove that as the approximations converge they cannot escape to infinity.
- For $K \subseteq \mathbb{C}$, need also toric varieties.

Abelian groups

1. Why are there no exponential maps in positive characteristic?
2. Prove the 6 equivalent forms of full kernel from slide 25

Z-group domains

In $\mathbb{C}_{\exp}$, the formula $\varphi(x)$ given by $\forall y[e^y = 1 \to e^{xy} = 1]$ defines $\mathbb{Z}$.
In $\mathbb{C}_{\exp}$, the sentence ψ given by $\exists t \forall x[\varphi(x) \iff e^{tx} = 1]$ is true. (Think $t = 2\pi i$.)
1. Show that in any exponential field F, $\varphi(x)$ defines a subring of F, and in any ELA-field in which ψ is true it defines a subring whose additive group is elementarily equivalent to $\langle \mathbb{Z}, + \rangle$.
2. Is there an integral domain R whose additive group is elementarily equivalent to $\langle \mathbb{Z}, + \rangle$, and whose first order theory is model-theoretically tame? E.g. stable? NIP?

Free extensions

Prove that F^{EA} is unique up to isomorphism over F, where F is a partial E-field.

Easy quasiminimality of $\mathbb{C}$ with integer powers

Let $\mathbb{C}_{\mathbb{Z}\mathrm{IP}}$ be the structure $\langle \mathbb{C}; +, \cdot, -, 0, 1, \mathbb{Z}, p \rangle$ where $p : \mathbb{C} \times \mathbb{Z} \to \mathbb{C}$ is the function $(z, n) \mapsto z^n$.
Prove that $\mathbb{C}_{\mathbb{Z}\mathrm{IP}}$ is a quasiminimal structure.