Generic derivations

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Introduction

Joint work with Giuseppina Terzo

T model complete theory expanding the theory of fields of characteristic 0
T admits **generic derivations** if T + "δ is a derivation" has a model completion T^{δ}_{g}.

If T is **algebraically bounded**, then T admits generic derivations.
T^{δ}_{g} inherits model theoretic properties from T: NIP, simplicity, uniform finiteness, etc.
Open problem: imaginaries of T^{δ}_{g}?

Q. Which theories admit generic derivations?
A. Algebraic geometry in a definable way inside T.
How much the model theoretic properties of T are inherited by T^{δ}_{g}?

Notes

Analogous results for tuples of (non)-commuting derivations

Contents

Generic derivations

T model complete L-theory, expanding field of char 0

If T is not model complete, replace it with its Morleyzation.

δ new function symbol

$T^\delta = T$ plus:

Additivity $\delta(x+y) = \delta x + \delta y$

Leibniz Rule $\delta(xy) = x\delta y + y\delta x$

T admits **generic derivation** if: exists model companion/completion T^δ_g of T^δ

Examples

- ACF_0 [Robinson '59, Blum '68]
- RCF [Singer '78]
- Henselian valued fields [Point et al.]
- Model complete "large/ample" fields [Tressl '05]

Algebraically bounded structure

Definition (van den Dries '89)

$\mathbb{K}$ is **algebraically bounded** if (in a saturated extension):
the field-theoretic and the model-theoretic acl coincide.

Example

The previous ones (ACF_0, RCF, HVF, model complete large fields . . .) plus:

- RV-expansions of HVF,
- "curve-excluding" fields [Johnson-Ye '23],
- expansion of above examples by a generic set [CP'98] . . .

Theorem (F-T '24)

If T is algebraically bounded, then it admits generic derivation.

Similar results for tuples of commuting or non-commuting derivations.

Notes

- Main property: "additivity" of algebraic dimension.
- Equivalent definition: every definable function is piecewise algebraic

Axiomatizations

$\mathbb{K} \models T$. Two possible axiomatizations for T_g^δ.

Deep $X \subseteq K^{n+1}$ $\mathbb{K}$-definable. $\Pi_n(X)$ Zariski dense $\implies \exists a\ \nabla^n a \in X$.

Wide $X \subseteq K^n \times K^n$ $\mathbb{K}$-definable. $\Pi_n(X)$ Zariski dense $\implies \exists \bar{a}\ (\bar{a}, \delta\bar{a}) \in X$.

Notes

- Similar results for a generic tuple of non-commuting derivations.
- Significantly more complicate to axiomatize a generic tuple of-commuting derivations.
- A third axiomatization in the style of Pierce-Pillay '98 is possible, but more complicate
- If T is not algebraically bounded, the "Wide" properties characterizes existentially closed models, but it might not be first order.

The proof

Lemma

- *(K, δ) differential field*
- *$F \supseteq K$*
- *$\bar{a} \in F^l$ algebraically independent over K*
- *$\bar{b} \in F^l$.*

Then, exists derivation ε on F s.t.

1. *ε extends δ;*
2. *$\varepsilon\bar{a} = \bar{b}$.*

Moreover, such ε is unique on the algebraic closure of $F(\bar{a})$.

Notes

An important consequence: the class of models of T^δ has the **amalgamation property**

Elimination of quantifiers

We assume T algebraically bounded.

Theorem (Strong elimination of quantifiers)

Every L^δ-formula $\alpha(\bar{x})$ is equivalent, modulo T_g^δ, to a formula of the form $\beta(\nabla x)$, where β is an L-formula.

Stability

T algebraically bounded.

Theorem

1. *If T is stable, then T_g^δ is stable.*
2. *If T is dependent, then T_g^δ is dependent.*

Proof.

Use Elimination of Quantifiers, plus:

- T is **stable** iff every indiscernible sequence is totally indiscernible;
- T is **dependent** iff every indiscernible sequence does not alternate infinitely many times.

□

ω-stability

Theorem

T_g^δ is ω-stable iff $T = \mathrm{ACF}_0$.

Proof.

T is ω-stable iff it has countably many types over a countable model.

Theorem (Hrushovski '92)

T is algebraically bounded and strongly minimal iff $T = \mathrm{ACF}_0$

□

In the case of a generic tuple of derivations,
$T_g^{\bar{\delta}}$ is ω-stable $\iff$ $T = \mathrm{ACF}_0$ and the derivations commute.

Simplicity

Theorem

If T is simple, then T_g^δ is simple.

Proof.

1) Forking on T_g^δ:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\delta}_{B} C \iff \nabla A \mathop{\smile\!\!\!\!\!\!|}_{\nabla B} \nabla C.$$

Theorem

If p is an L-type over ∇C, $p \mathop{\smile\!\!\!\!\!\!|}_{\nabla B} \nabla C$, and $p \upharpoonright_{\nabla B}$ is realized by $\nabla \bar{a}$, then $\exists \bar{a}'$ s.t. $\nabla \bar{a}'$ realizes p.

2) $\mathop{\smile\!\!\!\!\!\!|}^{\delta}$ satisfies assumption of Independence Theorem [Kim-Pillay '98] □

Notes

- $\nabla a := \left(\delta^n a : n \in \mathbb{N}\right)$ is the "Jet" of a.
- The characterization on forking on T_g^δ holds when T is simple.
  We don't know what happens when T is not simple.
- We have a canonical independence relation on T:

$$A \underset{B}{\overset{\mathrm{a}}{\downarrow}} C \iff \mathrm{trdeg}(A/B) = \mathrm{trdeg}(A/BC)$$

inducing a canonical independence relation on T_g^δ:

$$A \underset{B}{\overset{\mathrm{a}\delta}{\downarrow}} C \iff \nabla A \underset{\nabla B}{\overset{\mathrm{a}}{\downarrow}} \nabla C.$$

Open core

T has a definable topology with some weak conditions:

- field topology
- every open set is Zariski dense.

Theorem

*T is the **open core** of T_g^δ: every T_g^δ-definable open set is already T-definable.*

Corollary

*If $\dim(\partial X) < \dim(X)$ for every T-definable set, then T_g^δ has **Elimination of Imaginaries modulo** T: every T_g^δ-imaginary is inter-definable with a T-imaginary.*

Proof. [Tressl] □

Elimination of Imaginaries

Conjecture

T_g^δ has elimination of imaginaries modulo T.

Theorem

If T is simple with elimination of imaginaries, and elimination of hyperimaginaries, and "inddependence over substructures", then T_g^δ has elimination of imaginaries modulo T.

There are no known examples of simple theories which do not eliminate HI.

Proof. [Hrushovski-Chatzidakis '99] □

Definable groups

Theorem (Pillay, Point, Rideau-Kikuchi '25)

Let G be T_g^δ-definable. Then, G can be definably embedded in a T-interpretable group.

Extending types

$p = L^\delta$-type of $\bar{a}/\mathbb{K}$
$p_\nabla = L$-type of $\nabla\bar{a}/\mathbb{K}$.

Question

1. p definable iff p_∇ definable.
2. p heir/coheir of $p\restriction_{\mathbb{F}}$ iff p_∇ heir/coheir of $p_\nabla\restriction_{\mathbb{F}}$.

Should be **true** when T is **dependent**,
false for **pseudo-finite** fields.

Monoids of derivations

Γ monoid with generators $\bar{\delta}$.
$T[\Gamma]$ extension of T saying:

1. $\bar{\delta}$ are derivations
2. if two words in $\bar{\delta}$ are equal in Γ, they are equal as functions.

Question

For which Γ, $T[\Gamma]$ has a model companion?

Daddy structures

T expands the theory of fields of characteristic 0.
We no longer assume that T is algebraically bounded.

Definition

The algebraic dimension of a set is the Zariski dimension of its Zariski closure.
T is daddy if it has "(uniformly) Definable Algebraic Dimension".

Theorem

T.f.a.e.:

1. *"Being Zariski dense" is uniformly definable;*
2. *T is daddy;*
3. *T^δ has a model companion T^δ_g;*
4. *T with k (non-commuting) derivations has a model companion.*

Notes

- Uniform definability of a function d with argument a set, means that if $(X_i : i \in I)$ is a definable family, then $(d(X_i) : i \in I)$ is a definable function.
- Uniform definability of a property P means that its characteristic function is uniformly definable.

Proof idea

Remark

1) For every **existential** L^δ-formula $\alpha(\bar{x})$ there exists a quantifier-free L-formula $\beta(\bar{x}, \bar{x}', \bar{y}, \bar{y}')$ s.t.

$$T^\delta \models \alpha(\bar{x}) \iff \exists \bar{y} \beta(\bar{x}, \delta\bar{x}, \bar{y}, \delta\bar{y}).$$

2) If T is algebraically bounded, then for every **quantifier-free** L^δ-formula $\alpha(\bar{x})$ there exists a quantifier-free L-formula $\beta(\bar{x}, \bar{x}')$ s.t.

$$T^\delta \models \alpha(\bar{x}) \iff \beta(\bar{x}, \delta\bar{x}).$$

Corollary

$\langle M, \delta \rangle$ is existentially closed iff it satisfies

Wide *$X \subseteq K^n \times K^n$ $\mathbb{K}$-definable. $\Pi_n(X)$ Zariski dense $\implies \exists \bar{a}\ (\bar{a}, \delta\bar{a}) \in X$.*

Theorem

Assume that T is daddy and every T-definable function is an L-term.
Then, T^δ_g has elimination of quantifiers.

Uniform bounds

Theorem (van den Dries-Schmidt '84)

Given $d, n \in \mathbb{N}$, there exists a ***uniform bound*** *$b := b(d, n)$ s.t.. Let:*

K field,

I ideal in $K[X_1, \ldots X_n]$ generated by polynomials of degree $\leq d$.

Then its radical $J := \mathtt{rad}(I)$ is generated by polynomials of degree at most b.
Moreover, $J^b \subseteq I$ and I has at most b distinct minimal overprimes, all of which are generated by polynomials of degree $\leq b$.

Notes

The theorem was explcitely formulated in [Schoutens '10]

Geometry

$X \subseteq K^n$;
$I_K(X)$ is the ideal of X inside $K[\bar{x}]$.
$J < K[\bar{x}]$;
$V_K(J)$ is the zero set of J inside K^n.
K-radical of J is $K\text{-rad}(J) := I_K(V_K(J))$.

Uniform bounds for $I_K(X)$ and on $K\text{-rad}(J)$?

Example

- If K is real closed, K-rad is the real radical.
- If K is p-adically closed, K-rad is the p-adic radical.

In the above examples uniform bounds are known.

Notes

There are some results also for large fields [Sander '96]

Uniform bounds in daddy structures

Theorem

Assume that T is daddy. Then:

1. T_g^δ *is also daddy.*
2. *For every definable family of ideals* (J_t), $(K\text{-rad}(J_t))$ *is also a definable family of ideals.*
3. *Irreducible components are uniformly definable.*

Examples

How do we show that a structure is daddy?

Theorem

T is daddy iff $T^{\bar{\delta},c}$ (the expansion of T by k commuting derivations) and $T^{\bar{\delta},nc}$ (the expansion of T by k non-commuting derivations) have model companions/completions $T_g^{\bar{\delta},nc}$ and $T_g^{\bar{\delta},c}$.

Corollary

$T_g^{\bar{\delta},nc}$ and $T_g^{\bar{\delta},c}$ are also daddy.

Proof.

T_g^δ is daddy if $(T_g^\delta)^{\delta_2}$ has a model companion.
The latter is equal to the model companion of $T^{\delta_1,\delta_2,nc}$, which exists. □

Notes

- Establishing uniform or *effective* bounds in differential algebra is an active area of research
- Only partial results for differential ideals [Harrison-Trainor, Klys, Moosa '11]

Let $P(x)$ be a new unary predicate and $T^P := T$ as an $L(P)$-theory.

Theorem (Winkler '75, Chatzidakis-Pillay '98)

T^P has a model companion T^P_g iff T is Uniformly Finite.

Corollary

If T is daddy, then T^P_g is daddy.

Proof.

The model companion of $(T^\delta_g)^P$ (which exists because T^δ_g is daddy) is the model companion of $(T^P_g)^\delta$. □

Questions

Assume that T is daddy.

Question

Do stability/dependence/simplicity transfer to T^δ_g?

It is **not** true: for every L^δ-formula $\alpha(\bar{x})$ there exists and L-formula $\beta(\bar{x}, \bar{x}')$ s.t.
$T^\delta_g \models \alpha(\bar{x}) \iff \beta(\bar{x}, \delta\bar{x})$.