Definable rings in o-minimal structures

Annalisa Conversano

Massey University - Auckland, New Zealand

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o-minimality and objects over the real field

Definable groups in o-minimal structures $\leftrightarrow$ Lie groups

Forthcoming book:

- A. Conversano & A. Onshuus, *Lie groups and o-minimality*

  London Mathematical Society Lecture Notes Series, 2026.

Definable rings in o-minimal structures $\leftrightarrow$ real associative algebras

Fixing the terminology

- By **ring** I mean an abelian group with an associative multiplication that distributes over the group operation on the left and on the right. In particular, rings are not assumed to be unital nor commutative.

- An **associative algebra** over a field K is a vector space over K

$$(A, +, 0, a \mapsto ka)_{k \in K}$$

with an associative multiplication $\cdot$ such that:

– $(A, +, 0, \cdot)$ is a ring,

– the ring multiplication is compatible with the scalar multiplication:

$$\forall\, a, b \in A \text{ and } \forall\, k, s \in K \qquad (ka) \cdot (sb) = (ks)(a \cdot b)$$

If A is a unital n-dim ass. K-algebra, then $A \hookrightarrow M_n(K)$.

If A is a non-unital n-dim ass. K-algebra, then $A \hookrightarrow M_{n+1}(K)$.

In any case, every finite-dimensional associative algebra over K is K-definable (using a fixed basis as parameters)

Previous work on definable rings in o-minimal structures

Theorem (Pillay - 1988)

F ∞ def. field $\implies$ either $\dim F = 1$ & F rcf or $\dim F = 2$ and $F = K(\sqrt{-1})$, K rcf

Theorem (Otero, Peterzil, Pillay - 1996)

Let R be a definable ring in the o-minimal $\mathcal{M}$. Then

- R admits a definable ring-manifold structure;
- if $\mathcal{M}$ expands a rcf K,

  R admits a definable m-differentiable ring-manifold structure for any $m \in \mathbb{N}$,

  $\mathrm{Ann}_{L/R}(R) = \{0\} \implies R$ is an associative algebra over K.

Theorem (Peterzil, Steinhorn - 1999)

If a def. conn. ring R has no zero divisor, then R is a division ring & there is a def. rcf K s.t.

$$R = K \quad \text{or} \quad R = K(\sqrt{-1}) \quad \text{or} \quad R = \mathbb{H}(K)$$

The definably connected component of zero R^0 has finite index in R

$(R^0, +)$ splits in $(R, +)$, but $(R^0, +, \cdot)$ may not split in $(R, +, \cdot)$.

Example: Let $\mathcal{A} = (A, <, +, 0)$ be and ordered divisible abelian group. Fix $a \in A$, $a > 0$. On $[0, a[$ let $\oplus_a$ be the addition modulo a. Set $R = \mathbb{F}_2 \times [0, a[$ with operations

$$(t, x) \oplus (s, y) = (t + s,\ x \oplus_a y)$$

$$(t, x) \otimes (s, y) = \begin{cases} (0, a/2) & \text{if } t = s = 1 \\ (0, 0) & \text{otherwise.} \end{cases}$$

Then $(R, \oplus, \otimes)$ is a $\mathcal{A}$-definable ring. The definably connected component

$$R^0 = \{0\} \times [0, a[$$

has two complements in $(R, \oplus)$: $G_1 = \{(0, 0), (1, 0)\}$ and $G_2 = \{(0, 0), (1, a/2)\}$, but neither is a subring. Note that the finite ring

$$F = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & x & 0 \end{pmatrix} : x, y \in \mathbb{F}_2 \right\}$$

embeds in $(R, \oplus, \otimes)$ as a 2-torsion ideal with non-trivial intersection with $R^0 \subseteq \mathrm{Ann}(R)$.

Main results

Theorem

Every definably connected ring that is not a null ring defines an infinite field.

Theorem

Let R be a definably connected ring. Then R is a direct product of rings

$$R = R_0 \times R_1 \times \cdots \times R_s$$

$$R_0 \subseteq \mathrm{Ann}(R), \quad i > 0, R_i \text{ non-trivial ass. algebra over a } \textbf{def.} \text{ rcf } K_i$$

If $\mathcal{N}(R, +)$ is a direct sum of **definable** 1-dim subgroups,
then each R_i is a **definable** vector space over K_i and a **definable** ass. algebra.

Definability issues 1

Theorem

Let R be a definably connected ring. Then R is a direct product of rings

$$R = R_0 \times R_1 \times \cdots \times R_s$$

$R_0 \subseteq \mathrm{Ann}(R)$, $\quad i > 0$, R_i non-trivial ass. algebra over a **def.** rcf K_i

There are definable rings R such that $R_0 \subseteq \mathrm{Ann}(R)$ **cannot** be chosen to be definable:

Example. Let $\mathcal{M}$ be the real field. Set $R = \mathbb{R}^2 \times [1, e[$ with ring operations

$$(a, x, u) \oplus (b, y, v) = \begin{cases} (a+b, x+y, uv) & \text{if } uv < e \\ (a+b, x+y+1, uv/e) & \text{otherwise.} \end{cases}$$

$$(a, x, u) \otimes (b, y, v) = (0, ab, 0).$$

Then $R_1 = \mathbb{R}^2 \times \{1\}$ is a definable associative algebra, its additive complements are contained in $\mathrm{Ann}(R)$ and none of them is definable in $\mathcal{M}$.

Definability issues 2

If there is a 2-dimensional abelian torsion-free definable group that is not a direct sum of **definable** 1-dimensional subgroups, it should be possible to build associative K-algebras that are definable rings, but not definable K-vector spaces:

Example. Let $(K, +, \cdot)$ be a real closed field definable in the o-minimal structure $\mathcal{M}$. Suppose $f\colon K \times K \to K$ is a definable 2-cocycle with the respect to $+$. Define on K^2:

$$(x, a) \oplus (y, b) = (x + y, a + b + f(x, y))$$

$$(x, a) \otimes (y, b) = (0, xy).$$

Then $R = (K^2, \oplus, \otimes)$ is a $\mathcal{M}$-definable ring. Since $(K, +)$ is divisible, we know that f is a coboundary, the definable group $(K^2, \oplus)$ is isomorphic to $(K, +)^2$ and R is isomorphic to

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ a & x & 0 \end{pmatrix} : x, a \in K \right\}.$$

However, we do not know whether there is a definable 2-cocycle that is not *definably* coboundary, meaning that none of the complements of the 1-dimensional definable subgroup $\{0\} \times K$ in $(K^2, \oplus)$ is definable.

If there is such "bad" cocycle, the corresponding ring R is a definable ring and it is an associative K-algebra, but the scalar multiplication $K \times R \to R$ is not definable.

A natural strategy

If we are able to prove that R is a definable K-vector space, we are done:

Proposition

Let K be a definable rcf. If R is a definable ring & a definable K-vector space, then R is a (definable) associative K-algebra.

Proof.

Note: $G = (K, +)$ has only the two trivial definable subgroups.

Fix $u, v \in R$ and set

$$A = \{s \in K : u \cdot (sv) = s(u \cdot v)\}.$$

A is a def. subgroup of G and $1 \in A \implies A = G$.

Now fix $s \in K$ and set

$$B = \{r \in K : (ru) \cdot (sv) = (rs)(u \cdot v)\}.$$

Again, B is a def. subgroup of G and $1 \in B \implies B = G$.

□

However, we know from the previous example this cannot be done in general.

We need different strategies for different kind of rings.

Nilpotent rings & a structure theorem

Theorem

Let R be a n-dimensional def. conn. ring and $J(R)$ its Jacobson radical. TFAE:

1. R is a nilpotent ring.
2. R is a nil ring.
3. $R^{n+1} = \{0\}$.
4. $R = J(R)$.
5. Zero is the only idempotent element in R.

$$J(R) = \{a \in R : \forall\, r \in R\ \exists\, b \in R \text{ such that } bra - ra - b = 0\}.$$

If R is not nilpotent, $J(R)$ is the maximal nilpotent R-definable ideal,
it contains every nilpotent (definable or not) ideal of R,
the quotient $R/J(R)$ is a semiprime ring, and there is a definable subring S which is definably isomorphic to $R/J(R)$ s.t.

$$R = J(R) \oplus S.$$

Semiprime rings

R is **semiprime** iff $\{0\}$ is the only nilp. ideal. R is **simple** iff $\{0\}$ & R are the only ideals.

Theorem

Let R be a definably connected ring. The following are equivalent:

(i) R is semiprime.

(ii) R is semisimple.

(iiii) $J(R) = \{0\}$.

(iv) R is a direct product of simple definable rings.

(v) R is unital and either R is a division ring or there is a (unique) finite set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ such that $1 = \sum_{i=1}^{n} e_i$ and

$$R = \bigoplus_{i=1}^{n} Re_i = \bigoplus_{i=1}^{n} e_i R = \bigoplus_{i,j=1}^{n} e_i Re_j,$$

where $\{Re_i : i = 1, \ldots n\}$ is the set of minimal left ideals of R and $\{e_i R : i = 1, \ldots n\}$ is the set of minimal right ideals of R. Moreover, each $e_i Re_j$ is an infinite subring that is a division ring when $i = j$ and it is a null ring when $i \neq j$.

R is **semisimple** iff R is Artinian & $J(R) = \{0\}$.

Semiprime rings

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where $\{Re_i : i = 1, \ldots n\}$ is the set of minimal left ideals of R and $\{e_i R : i = 1, \ldots n\}$ is the set of minimal right ideals of R. Moreover, each $e_i R e_j$ is an infinite subring that is a division ring when $i = j$ and it is a null ring when $i \neq j$.

Example: $M_n(\mathbb{R}) \times M_k(\mathbb{C})$. The primitive orthogonal idempotents from (v) are the canonical bases of the two direct factors. This is always the case by the following:

Simple rings

Theorem

Let R be an infinite definable ring. The following are equivalent:

(a) R is simple.

(b) R is definably simple.

(c) R is definably connected and prime.

(d) There is a definable rcf K and a definably connected K-definable division ring D such that R is definably isomorphic to $M_n(D)$, for some $n \geqslant 1$.

R is **semiprime** iff $\{0\}$ is the only nilp. ideal.

R is **prime** iff for all ideals I and J of R

$$IJ = 0 \quad \Rightarrow \quad I = 0 \quad \text{or} \quad J = 0.$$

prime $\Rightarrow$ semiprime: if $I \neq \{0\}$ & $I^k = \{0\}$, set $J = \langle I^{k-1} \rangle$.

Unital rings

Given **any** subset X of R, $R(X)$ denotes the smallest definable subring of R containing X, that we know exists by DCC on definable subgroups.

Theorem

Let R be a definably connected ring. Then R is a direct product of rings

$$R = R_0 \times R_1 \times \cdots \times R_s$$

$$R_0 \subseteq \mathrm{Ann}(R), \quad i > 0, R_i \text{ non-trivial ass. algebra over a def. rcf } K_i$$

If R is unital, then:

- $R(1) = K_1 \times \cdots \times K_s$
- Each R_i is a definable K_i-vector space and a definable K_i-algebra.

Consequences:

If R is unital, every ideal is definable & R is Artinian and Noetherian.

The definable unitazation $R^\wedge$

Theorem

Let R be a definably connected ring. Then R is a direct product of rings

$$R = R_0 \times R_1 \times \cdots \times R_s$$

$$R_0 \subseteq \mathrm{Ann}(R), \quad i > 0, R_i \text{ non-trivial ass. algebra over a } \textbf{def.} \text{ rcf } K_i$$

if R is not unital, there is a unital definable ring containing R as an ideal iff $(R_0, +)$ is definable torsion-free and $(J(R), +)$ is a direct sum of definable 1-dimensional subgroups

$$J(R) = \bigoplus_{i=1}^{\dim J(R)} A_i,$$

where each $(A_i, +)$ admits a definable multiplication making it a real closed field. When this is the case, there is a smallest definably connected unital ring $R^\wedge$ containing R as an ideal, and $\dim R^\wedge - \dim R \leqslant \dim R_0 + s$.
We call $R^\wedge$ the **definable unitazation** of R.

The field case

Remarkably, the $\mathcal{M}$-definable rings coincide with the K-definable rings for any o-minimal expansion $\mathcal{M}$ of a rcf K (apart from the additive group):

Let $\mathcal{R} = (R, \oplus, \otimes)$ be a def. ring in an o-minimal expansion $\mathcal{M}$ of a rcf K. Then:

(a) either $\mathcal{R}$ is a finite-dimensional associative K-algebra;

(b) or $R^0 \subseteq \mathrm{Ann}(R)$ & there is a finite subring F of $\mathcal{R}$ such that $x \otimes y = 0$ whenever $x \notin F$ or $y \notin F$;

(c) or $\mathcal{R}$ is a direct product of rings

$$A \times B$$

where A and B are as in (a) & (b) respectively.

If $\mathcal{R}$ is unital, then A is unital & $B = F$ is a finite unital ring.

Question: Can the o-minimality assumption be weakened?

Finding the fields: nilpotent case - ad hoc construction

Finding the fields: semiprime case - using idempotents

Sketch of the general case

Open issues

- We know that given an infinite field K, finite-dimensional associative algebras over K and finite rings are K-definable. For which K and expansions $\mathcal{M}$ of K the $\mathcal{M}$-definable rings reduce to those? (as for rcf K and o-minimal expansions?)
- More generally, similar strategies could be used to study definable rings in other structures/theories.
- The Jacobson radical $J(R)$ is always R-definable and can be a general starting point. In "good" settings, it is likely to be nilpotent.
- Main o-minimal tools used here:
  - A good notion of dimension.
  - Finite n-torsion additive subgroup.
  - DCC on definable subgroups (possibly not essential).
  - The additive group of a rcf has only the trivial definable subgroups.
- Main tools from ring theory used here (combined with definability are very useful):
  - Brauer's Lemma.
  - Pierce decomposition (in the presence of idempotents).